XXVI. Communications Systems Research: Communication and Tracking

A. Performance of a Class of Q-Orthogonal Signals for Communication Over the Gaussian Channel

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1. The Signal Set and Optimal Receiver

Consider the set of QM signals:

$$x_m^{(q)}(t)\,(m = 1, 2, \cdots, M; q = 0, 1, 2, \cdots, Q-1)(0 \leqq t \leqq T), \tag{1}$$

all of which have the same energy ST and whose normalized inner products are

$$\rho_{mn}^{qr} = \frac{1}{ST}\int_0^T x_m^{(q)}(t)\, x_n^{(r)}(t)\, dt = \begin{cases} 0 \text{ for all } q, r & \text{if } m \neq n \\ \cos\dfrac{2\pi(q-r)}{Q} & \text{if } m = n \end{cases} \tag{2}$$

Examples of such signal sets are

$$x_m^{(q)}(t) = (2S)^{1/2}\sin\left(\frac{2\pi mt}{T} + \frac{2\pi q}{Q}\right)(0 \leqq t \leqq T) \tag{3a}$$

$$x_m^{(q)}(t) = s_m(t)\sin\left(\omega_0 t + \frac{2\pi q}{Q}\right)(0 \leqq t \leqq T) \tag{3b}$$

where $s_m(t)$ is a set of M equal energy orthogonal signals and ω_0 is a high enough frequency that the integral of the double frequency terms is negligible; or where $s_m(t)$ is a set of two-level orthogonal signals each consisting of M symbols and $\omega_0 = 2\pi M/T$.

Other examples are the algebraically generated polyphase code signals of Ref. 1.[1] We shall call this class of signals Q-orthogonal, since it is an obvious generalization of the class of biorthogonal signals, which corresponds to $Q = 2$.

Clearly the bandwidth occupancy (Ref. 2, pp. 3–12) of any signal set in this class is:

$$W = M/T \text{ cps}, \tag{4}$$

while the transmission rate in bits per second, if all signals are equiprobable, is

$$R = \log_2(MQ)/T. \tag{5}$$

The parameter normally constrained in a digital communication system is the bandwidth-to-rate ratio

$$\frac{W}{R} = \frac{M}{\log_2 M + \log_2 Q}. \tag{6}$$

[1]Also to be presented at Symposium on Time Varying Channels, June 5 through 9, 1965, Boulder, Colorado.

The optimum receiver, when the signal set consists of MQ equal energy equiprobable signals in the presence of additive white Gaussian noise, consists of a set of correlations and a comparator and decision device which decides in favor of the signal which maximizes

$$\int_0^T y(t)\, x_m^{(q)}(t)\, dt \tag{7}$$

where $y(t)$ is the received waveform during the given interval. It is well known that the performance of the communication system depends only on the ratio of the energy ST to the noise (one-sided) spectral density, N_0 and the $(MQ/2)$ inner products ρ_{mn}^{qr}. Thus, without loss of generality, we may assume henceforth that the signal set (3a) is used. Then

$$\int_0^T y(t)\, x_m^{(q)}(t)\, dt = (2S)^{1/2} \int_0^T y(t) \sin\left(\frac{2\pi mt}{T} + \frac{2\pi q}{Q}\right) dt$$

$$= \hat{y}_m \cos\frac{2\pi q}{Q} + \tilde{y}_m \sin\frac{2\pi q}{Q}$$

$$(m = 1, 2, \cdots, M; \quad q = 0, 1, 2, \cdots, Q-1) \tag{8}$$

where

$$\hat{y}_m = (2S)^{1/2} \int_0^T y(t) \sin\frac{2\pi mt}{T}\, dt \tag{8a}$$

and

$$\tilde{y}_m = (2S)^{1/2} \int_0^T y(t) \cos\frac{2\pi mt}{T}\, t\, dt \tag{8b}$$

or

$$\int_0^T y(t)\, x_m^{(q)}(t)\, dt = \left(\hat{y}_m^2 + \tilde{y}_m^2\right)^{1/2} \cos\left(\frac{2\pi q}{Q} - \tan^{-1}\frac{\tilde{y}_m}{\hat{y}_m}\right)$$

$$(m = 1, 2, \cdots, M; \quad q = 0, 1, 2, \cdots, Q-1). \tag{9}$$

Thus, it is clear that only $2M$ analog correlators, which generate the quantities of Eqs. (8a) and (8b), are necessary and that the remainder of the computation which leads to Eq. (9) can be performed by a special purpose digital computer.

2. A Sub-Optimal Receiver and Its Performance

To obtain a closed-form expression for the performance of the optimal receiver is a formidable and seemingly insurmountable task (Ref. 1). However, the following sub-optimal (but possibly nearly optimal) receiver is suggested by Eq. (8). The decision consists of two steps:

(1) Choose m so as to maximize $(\hat{y}_m^2 + \tilde{y}_m^2)^{1/2}$.

(2) With this value of m, choose q so as to maximize

$$\cos\left(\frac{2\pi q}{Q} - \tan^{-1}\frac{\tilde{y}_m}{\hat{y}_m}\right),$$

or equivalently to minimize

$$\left(\frac{2\pi q}{Q} - \tan^{-1}\frac{\tilde{y}_m}{\hat{y}_m}\right).$$

Such a receiver is easily mechanized by the parallel combination of M devices, one of which is shown in Fig. 1.

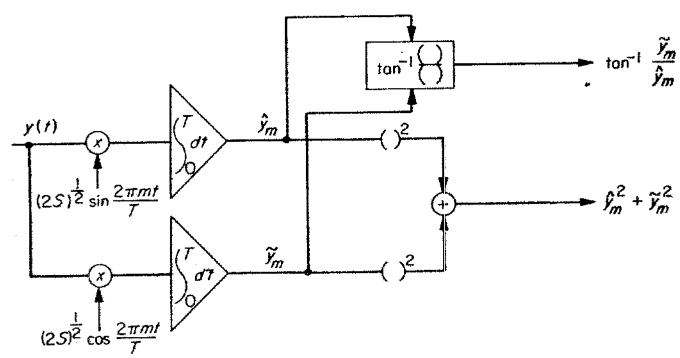

Fig. 1. An element of the sub-optimal receiver

The receiver bank resembles the optimal system for noncoherent reception. In fact, the first step in the procedure is exactly the same. The second step consists essentially of a phase estimation of the received phase angle corresponding to the largest m, comparison with each possible transmitted phase, and selection of the nearest one as the most likely. It is clear that if the signal set (3b) were used instead of (3a), the local signal inputs to the multipliers would be replaced by $s_m(t) \sin \omega_0 t$ and $s_m(t) \cos \omega_0 t$.

The error probability for this sub-optimal receiver (which, therefore, must be an upper bound on the error probability for the optimal receiver) is readily obtained. Without loss of generality, we may assume that signal

$x_1^{(0)}(t)$ was transmitted, since the signal set exhibits total symmetry. Thus, the probability of correct detection is

$$P_c = \text{Prob}\,(r_1 = \max_m r_m\,,\quad |\theta_1| < \pi/Q) \qquad (10)$$

where

$$r_m = (\hat{y}_m^2 + \tilde{y}_m^2)^{1/2}$$

and

$$\theta_m = \tan^{-1}\left(\frac{\tilde{y}_m}{\hat{y}_m}\right).$$

But since $x_1^{(0)}(t)$ is assumed sent and the additive noise, $n(t)$ is a zero-mean white Gaussian process of one-sided spectral density, N_0,

$$E(\hat{y}_m) = (2S)^{1/2} E \int_0^T y(t) \sin\frac{2\pi m}{T} t\, dt$$

$$= (2S)^{1/2} E \int_0^T \left[(2S)^{1/2} \sin\frac{2\pi t}{T} + n(t) \right]$$

$$\times \sin\frac{2\pi m}{T} t\, dt = \begin{cases} ST & (m = 1) \\ 0 & (m \neq 1) \end{cases}$$

$$E(\tilde{y}_m) = (2S)^{1/2} E \int_0^T y(t) \cos\frac{2\pi m}{T} t\, dt$$

$$= (2S)^{1/2} E \int_0^T \left[(2S)^{1/2} \sin\frac{2\pi t}{T} + n(t) \right]$$

$$\times \cos\frac{2\pi m}{T} t\, dt = 0$$

$$\text{Var}(\hat{y}_m) = 2S\, E \int_0^T \int_0^T n(t) \sin\frac{2\pi mt}{T} n(u) \sin\frac{2\pi mu}{T} du\, dt$$

$$= \int_0^T \int_0^T \frac{N_0}{2} \delta(t-u) \sin\frac{2\pi mt}{T} \sin\frac{2\pi mu}{T} du\, dt$$

$$= \frac{N_0 ST}{2}.$$

Similarly,

$$\text{Var}(\tilde{y}_m) = \frac{N_0 ST}{2}$$

and

$$\text{Cov}(\hat{y}_m, \tilde{y}_n) = 0 \quad (m = 1, 2, \cdots, M;\ n = 1, 2, \cdots, M).$$

Thus, the variables r_m are mutually independent, and all except r_1 have identical Rayleigh probability density functions. From Eq. (7) we have, after appropriate transformations (Ref. 3)

$$P_c = \int_{-\pi/Q}^{\pi/Q} d\theta_1 \int_0^\infty p(\theta_1, r_1)\, dr_1 \prod_{m=2}^{M} \int_0^{r_1} p(r_m)\, dr_m \qquad (11)$$

where

$$p(r_m) = \begin{cases} \dfrac{2r_m}{N_0 ST} \exp\left(-\dfrac{r_m^2}{N_0 ST}\right) & \text{for } r_m > 0 \\ 0 & \text{for } r_m < 0 \end{cases}$$

$$p(\theta_1, r_1) = \begin{cases} \dfrac{r_1}{\pi N_0 ST} \exp\left[-\dfrac{r_1^2 + (ST)^2 - 2STr_1 \cos\theta_1}{N_0 ST}\right] \\ (\text{for } r_1 \geq 0 \text{ and } -\pi \leq \theta_1 < \pi). \\ 0 \quad \text{otherwise} \end{cases}$$

Thus,

$$P_c = \int_{-\pi/Q}^{\pi/Q} d\theta_1 \int_0^\infty \frac{r_1}{\pi N_0 ST} \exp\left[-\frac{r_1^2 + (ST)^2 - 2STr_1\cos\theta_1}{N_0 ST}\right] \left[1 - \exp\frac{-r_1^2}{N_0 ST}\right]^{M-1} dr_1$$

$$= \frac{ST}{\pi N_0} \int_{-\pi/Q}^{\pi/Q} d\theta \int_0^\infty r \exp\left[-\left(\frac{ST}{N_0}\right)(r^2 + 1 - 2r\cos\theta)\right] \left[1 - \exp\left(-\frac{r^2 ST}{N_0}\right)\right]^{M-1} dr$$

$$= \frac{ST}{\pi N_0} \int_{-\pi/Q}^{\pi/Q} d\theta \int_0^\infty r \exp\left[\left(\frac{ST}{N_0}\right)(r^2 + 1 - 2\cos\theta)\right] \sum_{k=0}^{M-1} (-1)^k \binom{M-1}{k} \exp\left(\frac{-kr^2 ST}{N_0}\right) dr$$

$$= \sum_{k=0}^{M-1} \frac{(-1)^k \binom{M-1}{k} \exp\left(\frac{-\alpha k}{k+1}\right)}{k+1}$$

$$\times \int_{-\pi/Q}^{\pi/Q} \frac{\exp\left(\frac{-\alpha}{k+1}\right)}{2\pi} \left[1 + \left(\frac{\pi\alpha}{k+1}\right)^{1/2} \cos\theta \left\{1 + \text{erf}\left(\left[\frac{\alpha}{k+1}\right]^{1/2} \cos\theta\right)\right\} \exp\left(\frac{\alpha\cos^2\theta}{k+1}\right)\right] d\theta \qquad (12)$$

where $\alpha = ST/N_0$. The integral in the kth term of the sum is the probability distribution $P(|\theta| < \pi/Q)$ of the phase angle of a sinusoid in additive Gaussian noise for signal-to-noise ratio $\alpha/(k+1)$ (Ref. 3).

A lower bound on P_c (and consequently on upper bound on the error probability P_E) is obtained by neglecting all but the first two terms. The first term by itself is an upper bound on P_c (lower bound on P_E) for this sub-optimal receiver. In the next part of this article, we show that this is a lower bound on P_E also for the optimal receiver.

3. *Bounds on Error Probability for the Optimal Receiver*

In Part *1*, we showed that the optimum receiver selects q and m so as to maximize Eq. (6). Without loss of generality, let us assume once again that $x_1^{(0)}(t)$ was transmitted. Now using the notation of Eq. (7), we see from (6) that if

$$r_1 > \max_{m \neq 1} r_m$$

but $|\theta_1| > \pi/Q$, we shall have the correct m but the wrong q. On the other hand, if

$$r_1 < r_\mu = \max_{m \neq 1} r_m,$$

we may still be correct if $r_1 \cos\theta_1 > r_m$, and $|\theta_1| < \pi/Q$. However, if $|\theta_1| > \pi/Q$, we shall always have an error. Thus, a lower bound on the error probability for the optimum receiver is

$$P_E > \mathrm{Prob}\left(|\theta_1| > \frac{\pi}{Q}\right). \tag{13}$$

The error probability for the sub-optimal receiver is clearly an upper bound to that for the optimal receiver. Thus,

$$\begin{aligned}\int_{|\theta|>\pi/Q} p(\theta,\alpha)\,d\theta &< P_E \\ &< 1 - \sum_{k=0}^{M-1} \frac{(-1)^k}{k+1}\binom{M-1}{k}\exp\left(\frac{-\alpha k}{k+1}\right)\int_{|\theta|<\pi/Q} p\left(\theta,\frac{\alpha}{k+1}\right)d\theta \\ &< \int_{|\theta|>\pi/Q} p(\theta,\alpha)\,d\theta + \frac{(M-1)}{2}\exp\left(\frac{-\alpha}{2}\right)\int_{|\theta|<\pi/Q} p\left(\theta,\frac{\alpha}{2}\right)d\theta\end{aligned} \tag{14}$$

where

$$p(\theta,\gamma) = \frac{\exp(-\gamma)}{2\pi}\{1 + (\pi\gamma)^{1/2}\cos\theta\,[1 + \mathrm{erf}(\gamma^{1/2}\cos\phi)]\exp(\gamma\cos^2\theta)\}$$

is the probability density function of the phase of the sum of a fixed phase sinusoid and Gaussian noise with SNR $= \gamma$.

Finally, using a result of Arthurs and Dym (Ref. 4), we may bound the integrals in Eq. (11) by

$$\frac{1}{2}\mathrm{erfc}\left(\gamma^{1/2}\sin\frac{\pi}{Q}\right) < \int_{|\theta|>\pi/Q} p(\theta,\gamma)\,d\theta < \mathrm{erfc}\left(\gamma^{1/2}\sin\frac{\pi}{Q}\right)$$

so that

$$\frac{1}{2}\mathrm{erfc}\left(\alpha^{1/2}\sin\frac{\pi}{Q}\right) < P_E < \mathrm{erfc}\left(\alpha^{1/2}\sin\frac{\pi}{Q}\right) + \frac{(M-1)}{2} \times \exp\left(\frac{-\alpha}{2}\right)\mathrm{erf}\left[\left(\frac{\alpha}{2}\right)^{1/2}\sin\frac{\pi}{Q}\right] \tag{15}$$

where

$$\alpha = \frac{ST}{N_0} = \frac{S}{N_0 R}\log_2(MQ)$$

and where $R = 1/T_b$ is the rate in bits per second. Letting

$$\beta = \frac{S}{N_0 R}\log_2 e,$$

$$M = \frac{1}{Q}\exp\left(\frac{k}{\beta\sin^2\pi/Q}\right)$$

and bounding the complementary error function by

$$\left(\frac{2}{\pi}\right)^{1/2}\frac{e^{-x^2/2}}{x}\left(1 - \frac{1}{x^2}\right) < \mathrm{erfc}\left(\frac{x}{2^{1/2}}\right) < \left(\frac{2}{\pi}\right)^{1/2}\frac{e^{-x^2/2}}{x} \tag{16}$$

yields the result that

$$\frac{e^{-k}}{2(\pi k)^{1/2}}\left(1 - \frac{1}{2k}\right) < P_E < \left(\frac{e^{-k}}{(\pi k)^{1/2}}\right) + \frac{e^{-k(\beta-2)/2\beta}}{2Q}. \tag{17}$$

Consequently, in order to achieve small error probabilities, it is imperative that k be reasonably large and, since

$$M = \frac{1}{Q}\exp\left(\frac{k}{\beta\sin^2\pi/Q}\right),$$

Q must be small compared to M. Note that if

$$\beta = \frac{S}{N_0 R} \log_e 2 > 2,$$

the upper and lower bounds on the error probability can be made arbitrarily small as k and Q are increased.

B. Another Look at the Optimum Design of Tracking Loops

R. C. Tausworthe

In 1955, Jaffe and Rechtin (Ref. 5) published the first sophisticated attempt at characterizing the optimum design of phase-locked loops. In the course of their work, they used an example which specified the transfer function of a loop best able to follow a frequency-step input insofar as minimizing transient error and phase noise are concerned. For simplicity, they assumed that the initial phase error was zero; the resulting filter function was one with one real zero and two complex poles, at a damping factor $\zeta = 0.707$, regardless of the initial frequency offset. The example was meant only to illustrate the optimization method, but since that time most systems have been designed using the parameters set by the example.

By using the same technique developed in the Jaffe–Rechtin paper, but assuming that the initial phase angle is random, a different result appears. Damping in the loop is always greater than $\zeta = 0.707$, and in all cases of practical interest, the system is *overdamped.* (Both poles lie on the negative real axis.)

Because the initial phase error is not generally known *a priori* (thus random), this latter design is one which seems to be of more practical use in most tracking applications.

1. Optimum Loops for Random Doppler Tracking

There are two sources of error during the initial acquisition of phase lock in a tracking receiver. First, there is a transient error as the system passes from its initial state to the steady-state tracking state. Second, there is phase jitter due to the presence of noise at the loop input. The technique developed by Jaffe and Rechtin was a Wiener optimization of the linearized loop, but with a constraint on the total mean-square transient error. Following this technique, the optimum loop transfer function was found to be specified by the formula

$$H_{opt}(s) = \frac{1}{[\Psi(s)]^+} \left[\frac{\lambda^2 \mathcal{E}\, D(s)\, D(-s)}{[\Psi(s)]^-} \right]_{pr}, \tag{1}$$

where

$\Psi(s) = \lambda^2 \mathcal{E}\, D(s)\, D(-s) + N_0/A^2$

$D(s)$ = doppler-phase Laplace transform

λ^2 = Lagrange multiplier (to be evaluated)

A^2 = loop input carrier power

N_0 = double-sided noise spectral density

$\mathcal{E}$ = expectation operator

$[\ ]^+$ = left half-plane "square-root" factorization operator

$[\ ]^-$ = right half-plane "square-root" factorization operator

$[\ ]_{pr} = \mathcal{L}\mathcal{F}^{-1}$, the physical-realizability operator

The reader is referred to Ref. 3 or 5 for further explanation of the operators above and for the development of Eq. (1).

The optimization of interest is concerned with finding $H_{opt}(s)$ when the input doppler $d(t)$ has the form

$$d(t) = \theta_0 + \omega_0 t, \tag{2}$$

where θ_0 is a uniformly distributed phase angle, and where ω_0 is a random variable whose mean-square value is Ω_0^2. The Laplace transform of $d(t)$ is

$$D(s) = \frac{\theta_0}{s} + \frac{\omega_0}{s^2}, \tag{3}$$

and hence the expected value of $D(s)\,D(-s)$ is

$$\mathcal{E}[D(s)\,D(-s)] = -\left(\frac{\pi^2}{3}\right)\frac{1}{s^2} + (\Omega_0^2)\left(\frac{1}{s^4}\right) \tag{4}$$

The first order of business is the factorization of $\Psi(s)$:

$$[\Psi(s)]^+ = \frac{N_0^{1/2}}{As^2}\left[s^4 + \left(\frac{\pi^2\lambda^2A^2}{3N_0}\right)s^2 + \frac{\lambda^2A^2\Omega_0^2}{N_0}\right]^+$$

$$= \frac{N_0^{1/2}}{As^2}\left[s^2 + \left(\frac{A^2\lambda^2\pi^2}{3N_0} + \frac{2\lambda A\Omega_0}{N_0^{1/2}}\right)^{1/2} s + \frac{\lambda A \Omega_0}{N_0^{1/2}}\right]. \tag{5}$$